ON THE STABILITY OF ROTATION OF A TOP WITH A CAVITY FILLED WITH A VISCOUS LIQUID

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In papers [1,2] there was given the formulation and the solution of the problem of the stability of relative rotational motion of a symmetrical rigid body with a cavity, partially or completely filled with an ideal liquid, with respect to parameters, characterizing the motion of the rigid body and the projections of the moment of momentum of the liquid.

In the present paper the problem of the stability of motion of an asymmetric heavy rigid body with one point fixed is solved in an analogous formulation. The body has a cavity, completely filled with a viscous liquid. Using the second method of Liapunov sufficient conditions are found for stability of rotation with respect to the vertical of the rigid body with a liquid.

1. Let $O\xi\eta\zeta$ be the fixed rectangular system of coordinates, having their origin at the fixed point, with the axis $O\zeta$ directed vertically upward. We introduce also a moving rectangular system of coordinates $Oxyz$, whose axes coincide with the principal axes of inertia of the rigid body at its fixed point O. The principal moments of inertia of the body with respect to the axes x, y, z are designated by A_1, B_1, C_1, respectively. Let M_1 be the mass of the body, x_1, y_1, z_1 the coordinates of its center of mass.

We assume that the body possesses a cavity of arbitrary shape; for the sake of simplicity we assume that the axes x, y, z are principal axes of inertia of the volume of the cavity. Let the cavity be filled completely by a homogeneous, incompressible, heavy viscous liquid.

Let M_2 designate the mass of the fluid, x_2, y_2, z_2 the coordinates of its center of mass, ρ the density, A_2, B_2, C_2 the moments of inertia of the fluid with respect to the axes x, y, z, μ the coefficient of viscosity, $\nu = \mu/\rho$ the kinematic coefficient of viscosity.

It is assumed that no external forces are acting on the rigid body and the liquid, with the exception of the forces of gravity and the reaction at the fixed point.

The rigid body and the liquid-filled cavity will be regarded as a single mechanical system, for the derivation of the equations of motion of which we shall use the theorem of angular momentum. The angular momentum of the system with respect to the fixed point is compounded geometrically from the angular momentum of the body G_1 and the angular momentum of the liquid G_2.

If p, q, r designate the projections on the moving axes of the vector of instantaneous angular velocity of the body, then the projections of the vector G_1 on these axes will be equal to A_1p, B_1q, C_1r, respectively.

Let us designate by v_x, v_y, v_z the projections on the moving axes of the velocity of the liquid in its motion with respect to the fixed axes $O\xi\eta\zeta$. If we introduce also the vector of the relative velocity of the liquid in its motion with respect to the axes $Oxyz$, whose projections on these axes are designated by u, v, w, then the following formulas will be valid:

$$v_x = qz - ry + u, \qquad v_y = rx - pz + v, \qquad v_z = py - qx + w \tag{1.1}$$

The projections of the moving axes of the vector G_2 which represents the angular momentum of the fluid are

$$G_{2x} = \rho \int_\tau (yv_z - zv_y)\, d\tau, \qquad G_{2y} = \rho \int_\tau (zv_x - xv_z)\, d\tau, \qquad G_{2z} = \rho \int_\tau (xv_y - yv_x)\, d\tau$$

where τ designates the volume of the liquid-filled cavity.

Using Formulas (1.1) we easily obtain

$$G_{2x} = A_2p + g_1, \qquad G_{2y} = B_2q + g_2, \qquad G_{2z} = C_2r + g_3 \tag{1.2}$$

where

$$g_1 = \rho \int_\tau (yw - zv)\, d\tau, \qquad g_2 = \rho \int_\tau (zu - xw)\, d\tau, \qquad g_3 = \rho \int_\tau (xv - yu)\, d\tau$$

designate the projections on the moving axes of the relative angular-momentum vector of the liquid.

The theorem of angular momentum of the system expressed in terms of projections on moving axes leads to the following equations of motion of the rigid body with a liquid-filled cavity:

$$
\begin{aligned}
A\frac{dp}{dt}+(C-B)qr+\frac{dg_1}{dt}+qg_3-rg_2&=Mg(z_0\gamma_2-y_0\gamma_3)\\
B\frac{dq}{dt}+(A-C)pr+\frac{dg_2}{dt}+rg_1-pg_3&=Mg(x_0\gamma_3-z_0\gamma_1)\\
C\frac{dr}{dt}+(B-A)pq+\frac{dg_3}{dt}+pg_2-qg_1&=Mg(y_0\gamma_1-x_0\gamma_2)
\end{aligned}
\qquad (1.3)
$$

Here

$$A=A_1+A_2,\qquad B=B_1+B_2,\qquad C=C_1+C_2$$

designate the moments of inertia of the system with respect to the moving axes, $M = M_1 + M_2$ is the mass of the system, g is the acceleration of gravity, x_0, y_0, z_0 are the coordinates of the center of gravity of the system, where

$$Mx_0=M_1x_1+M_2x_2,\quad My_0=M_1y_1+M_2y_2,\quad Mz_0=M_1z_1+M_2z_2$$

γ_1, γ_2, γ_3 designate the direction cosines of the axis $O\zeta$ with respect to moving axes which satisfy Poisson's equation

$$\frac{d\gamma_1}{dt}=r\gamma_2-q\gamma_3,\qquad \frac{d\gamma_2}{dt}=p\gamma_3-r\gamma_1,\qquad \frac{d\gamma_3}{dt}=q\gamma_1-p\gamma_2 \qquad (1.4)$$

To obtain the complete system of equations of motion, Equations (1.3), (1.4) must be supplemented by the Navier-Stokes equations for the motion of a viscous incompressible heavy liquid, with respect to moving axes, together with the incompressibility equation

$$
\begin{aligned}
&\frac{d}{dt}(u+qz-ry)+q(w+py-qx)-r(v+rx-pz)\\
&\qquad=-g\gamma_1-\frac{1}{\rho}\frac{\partial p_1}{\partial x}+\nu\Delta u\\
&\frac{d}{dt}(v+rx-pz)+r(u+qz-ry)-p(w+py-qx)\\
&\qquad=-g\gamma_2-\frac{1}{\rho}\frac{\partial p_1}{\partial y}+\nu\Delta v\\
&\frac{d}{dt}(w+py-qx)+p(v+rx-pz)-q(u+qz-ry)\\
&\qquad=-g\gamma_3-\frac{1}{\rho}\frac{\partial p_1}{\partial z}+\nu\Delta w\\
&\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}=0\qquad\left(\Delta=\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}\right)
\end{aligned}
\qquad (1.5)
$$

Here p_1 designates the hydrodynamic pressure.

The solutions of Equations (1.5) must satisfy at the walls S the boundary conditions

$$u = v = w = 0 \quad \text{on } S \tag{1.6}$$

while in the case of an ideal liquid only the normal component of the relative velocity of the liquid should be equal to zero at S.

We pass now to the derivation of certain relations which we will need in the sequel for the solution of the stability problem.

We first multiply Equations (1.3) by p, q, r, respectively, and add them; the first three equations (1.5) are multiplied by u, v, w, respectively, and are added; the result is multiplied by $\rho\, d\tau$ and integrated over the volume τ of the cavity and then added to the first sum.

Taking into account the incompressibility equation and the boundary conditions (1.6) for the liquid, as well as Poisson's Equations (1.4), we obtain after simple manipulation the following equation:

$$\frac{d}{dt}(T_1 + T_2 + V) = -\mu \int_\tau \left[\left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right)^2 + \left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)^2\right] d\tau \tag{1.7}$$

where $T = T_1 + T_2$ designates the kinetic energy of the system, the kinetic energies of the body and the liquid being, respectively,

$$2T_1 = A_1 p^2 + B_1 q^2 + C_1 r^2$$

$$2T_2 = A_2 p^2 + B_2 q^2 + C_2 r^2 + \rho \int_\tau (u^2 + v^2 + w^2)\, d\tau +$$

$$+ 2\rho \int_\tau [u\,(qz - ry) + v\,(rx - pz) + w\,(py - qx)]\, d\tau \tag{1.8}$$

and

$$V = Mg\,(x_0\gamma_1 + y_0\gamma_2 + z_0\gamma_3) \tag{1.9}$$

designating the potential energy of the gravity force.

From Equation (1.7) it follows that

$$T + V \leqslant T_0 + V_0 \tag{1.10}$$

where the subscript (0) designates the initial value of the corresponding quantity.

For an ideal liquid $\mu = 0$ and in relation (1.10) the equality sign will apply.

Let us now multiply Equations (1.3) by γ_1, γ_2, γ_3, respectively, add them, and obtain the integral of the areas, taking into account Equations (1.4)

$$(Ap + g_1)\gamma_1 + (Bq + g_2)\gamma_2 + (Cr + g_3)\gamma_3 = \text{const} \qquad (1.11)$$

It is also obvious that Equations (1.4) admit a first integral

$$\gamma_1^2 + \gamma_2^2 + \gamma_3^2 = 1 \qquad (1.12)$$

2. We shall now consider the case when the center of gravity of the system is located on the principal axis of inertia Oz of the rigid body and the liquid, i.e. when $x_0 = y_0 = 0$.

Equations (1.3) through (1.5) then admit the particular solution

$$p = q = 0, \quad r = \omega, \quad G_{2x} = G_{2y} = 0, \quad G_{2z} = G = C_2\omega \qquad (2.1)$$
$$\gamma_1 = \gamma_2 = 0, \quad \gamma_3 = 1, \quad u = v = w = 0$$

which describes a uniform rotation of the rigid body with a liquid-filled cavity about the vertical.

Let us study the stability of the unperturbed motion (2.1) with respect to the projections of the instantaneous angular velocity of the body p, q, r, the projections of the angular momentum vector of the liquid G_{2x}, G_{2y}, G_{2z} and the direction cosines of the vertical γ_1, γ_2, γ_3.

In the disturbed motion we set

$$r = \omega + \xi, \quad G_{2z} = G + \eta, \quad \gamma_3 = 1 + \zeta$$

while the remaining variables retain the earlier notation. If these values for the variables are substituted into Equations (1.3) through (1.5), we obtain the equations of disturbed motion; we will not write them down explicitly.

Before passing to the solution of the stability problem, we transform the kinetic energy expression of the liquid. During motion of the system the quantities p, q, r, g_1, g_2, g_3 will represent some functions of time.

In place of G_{2x}, G_{2y}, G_{2z} we introduce new independent functions of time ω_1, ω_2, ω_3 defined by the equations

$$\omega_1 = \frac{1}{A_2} G_{2x}, \quad \omega_2 = \frac{1}{B_2} G_{2y}, \quad \omega_3 = \frac{1}{C_2} G_{2z} \qquad (2.2)$$

If the velocities of the liquid v_x, v_y, v_z are known, the functions $\omega_i(t)(i = 1, 2, 3)$ will also be given. We also consider the quantities

$v_i(t, x, y, z)$, defining them by the formulas

$$v_1 = v_x + \omega_3 y - \omega_2 z,\ v_2 = v_y + \omega_1 z - \omega_3 x,\ v_3 = v_z + \omega_2 x - \omega_1 y \quad (2.3)$$

It is then easily seen that on the strength of Equations (2.2)

$$\rho \int_\tau (yv_3 - zv_2)\, d\tau = \rho \int_\tau (zv_1 - xv_3)\, d\tau = \rho \int_\tau (xv_2 - yv_1)\, d\tau = 0 \quad (2.4)$$

Conversely, if $\omega_i(t)$, $v_i(t, x, y, z)(i = 1, 2, 3)$ are known, then Formulas (2.3) permit the determination of v_x, v_y, v_z.

Using Formulas (2.3) and (2.4), the expression for the kinetic energy of the liquid may be represented now in the form

$$2T_2 = \frac{G_{2x}^2}{A_2} + \frac{G_{2y}^2}{B_2} + \frac{G_{2z}^2}{C_2} + \rho \int_\tau (v_1^2 + v_2^2 + v_3^2)\, d\tau \quad (2.5)$$

From here, among other things, there immediately follows the inequality

$$2T_2 S \geqslant G_{2x}^2 + G_{2y}^2 + G_{2z}^2, \qquad S = \max (A_2, B_2, C_2)$$

established by Liapunov and used in papers [1,2].

Let us note that the functions $\omega_i(t)$, introduced by Formulas (2.2), may be interpreted as projections on the axes x, y, z of the instantaneous angular velocity of such a rigid body with a fixed point O, which possesses the same shape as the liquid, consists of the same material particles as the liquid, and whose angular-momentum vector is geometrically equal to the angular-momentum vector of the liquid. The functions $v_i(t, x, y, z)$ may then be treated as the projections on the moving axes of the velocity vector of the liquid in its motion with respect to the rigid body [3].

Passing to the study of stability of the unperturbed motion (2.1), we note that there

$$\omega_1 = \omega_2 = 0, \qquad \omega_3 = \omega, \qquad v_1 = v_2 = v_3 = 0$$

For the perturbed motion of the system, on the strength of Equation (1.7), we will have

$$dV_1 / dt \leqslant 0 \quad (2.6)$$

where

$$V_1 \equiv A_1p^2 + B_1q^2 + C_1(2\omega\xi + \xi^2) + \frac{1}{A_2}G_{2x}^2 + \frac{1}{B_2}G_{2y}^2 + \\ + \frac{1}{C_2}(\eta^2 + 2C_2\omega\eta) + 2Mgz_0\zeta + \rho\int_\tau (v_1^2 + v_2^2 + v_3^2)\,d\tau \qquad (2.7)$$

It is also easily seen that the equations of perturbed motion admit the first integrals

$$V_2 = (A_1p + G_{2x})\gamma_1 + (B_1q + G_{2y})\gamma_2 + C_1\xi + \eta + C_1(\omega + \xi)\zeta + \\ + (C_2\omega + \eta)\zeta = \text{const} \qquad (2.8)$$
$$V_3 = \gamma_1^2 + \gamma_2^2 + \zeta^2 + 2\zeta = 0$$

Let us consider the function

$$V = V_1 - 2\omega V_2 + (C\omega^2 - Mgz_0)V_3 + \frac{1}{4}\mu V_3^2$$
$$= A_1p^2 - 2\omega(A_1p + G_{2x})\gamma_1 + \frac{1}{A_2}G_{2x}^2 + (C\omega^2 - Mgz_0)\gamma_1^2 +$$
$$+ B_1q^2 - 2\omega(B_1q + G_{2y})\gamma_2 + \frac{1}{B_2}G_{2y}^2 + (C\omega^2 - Mgz_0)\gamma_2^2 +$$
$$+ C_1\xi^2 - 2\omega(C_1\xi + \eta)\zeta + \frac{1}{C_2}\eta^2 + (C\omega^2 - Mgz_0 + \mu)\zeta^2 +$$
$$+ \frac{1}{2}\mu\zeta(\gamma_1^2 + \gamma_2^2 + \zeta^2) + \rho\int_\tau (v_1^2 + v_2^2 + v_3^2)\,d\tau \qquad (2.9)$$

where the constant $\mu > Mgz_0$.

In accordance with Sylvester's criterion of a positive-definite function V, it is necessary and sufficient to satisfy the following inequalities:

$$(C - A)\,\omega^2 - Mgz_0 > 0, \qquad (C - B)\,\omega^2 - Mgz_0 > 0$$

If one assumes without loss of generality that $A > B$, the second of these conditions is obviously satisfied if the first is satisfied:

$$(C - A)\,\omega^2 - Mgz_0 > 0 \qquad (2.10)$$

The derivative of the function V, taken for the equations of perturbed motion, will be nonpositive as a consequence of inequality (2.6).

Hence in condition (2.10) the function V satisfies all conditions of Liapunov's stability theorem, which proves the stability of the unperturbed motion (2.10) of a rigid body with a cavity, filled with a viscous liquid, with respect to the quantities p, q, r, G_{2x}, G_{2y}, G_{2z}, y_1, y_2, y_3, as well as v_1, v_2, v_3.

Thus it is proved that condition (2.10) is a sufficient stability condition of unperturbed motion (2.1) of a rigid body with a cavity filled with a viscous liquid.

Let us note that condition (2.10) is of the same form as the stability condition for rotation about the vertical of a heavy rigid body of weight Mg with moments of inertia $A = A_1 + A_2$ and $C = C_1 + C_2$

For a single elongated rigid body, for which $A_1 > C_1$, a condition of the form (2.10) is not satisfied. However, for a body with a cavity it may be satisfied for a sufficiently large angular velocity ω, if the shape of the cavity is chosen in such a manner that the inequalities are satisfied:

$$C_2 > A_2, \qquad C_1 + C_2 - A_1 - A_2 > 0$$

It is of interest to point out that the stability condition (2.10) does not depend at all on the viscosity of the liquid and is thus valid also for an ideal liquid.

In the case of a nonviscous liquid, and if at the initial instant the motion is irrotational or if the liquid is at rest, then, in accordance with Lagrange's theorem, the motion of the liquid will remain irrotational at all times.

The kinetic energy of the liquid will then equal

$$2T_2 = A_2^* p^2 + B_2^* q^2 + C_2^* r^2$$

where A_2^*, B_2^*, C_2^* designate the moments of inertia of an equivalent rigid body in the sense of Zhukovskii [4], whereby

$$A_2 > A_2^*, \qquad B_2 > B_2^*, \qquad C_2 > C_2^*$$

In the case of the rotational motion of an ideal liquid, the stability condition of an irrotational motion (2.1) will thus be of the form

$$(C_1 + C_2^* - A_1 - A_2^*)\,\omega^2 - Mgz_0 > 0 \tag{2.11}$$

We note that if the cavity is axially symmetric with respect to the axis Oz, then $C_2^* = 0$, and if condition (2.11) is satisfied, then condition (2.10) will also be satisfied provided the following inequality holds:

$$C_2 \geqslant A_2 - A_2^*$$

The stability of rotation of a rigid body with a cavity filled completely with an ideal liquid which is in a state of irrotational motion was first studied by Chetaev [5].

In the case of an axially symmetric cavity Chetaev obtained the necessary and sufficient stability condition of unperturbed motion with respect to the variables $p, q, r, \gamma_1 \ \gamma_2 \ \gamma_3$ in the form

$$C_1^2\omega^2 - 4(A_1 + A_2^*)Mgz_0 > 0$$

The problem of the stability of a symmetric top with an axially symmetric cavity filled completely with an ideal liquid which is in a state of vortex motion was studied by Sobolev in linear formulation [6].

In particular, Sobolev proved that if the following inequality is valid:

$$L = (C_1 + C_2 - A_1 - A_2) - \frac{Mgz_0}{\omega^2} > 0 \qquad (2.12)$$

then the operator e^{iBt}, characterizing the perturbed motion of the system, is bounded.

It is obvious that condition (2.12) on the boundedness of this operator coincides with the stability condition of unperturbed motion (2.1) with respect to $p, q, r, G_{2x}, G_{2y}, G_{2z}, \gamma_1, \gamma_2, \gamma_3$.

We note finally that in the case of inertial motion of a rigid body with a liquid about the center of gravity of the system ($z = 0$) condition (2.1) takes on the form

$$C - A > 0$$

Hence, the permanent rotations of a rigid body with a liquid-filled cavity about the small axis of the central ellipsoid of inertia of the system are stable.

This result may be considered as a certain supplement to the well-known theorem of Zhukovskii [4] on the motion of a rigid body completely filled with a viscous liquid.

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